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FIXED POINTS AND TORSION ON KÄHLER MANIFOLDS[1]

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1. Introduction and an example

When a 1-parameter group acts by isometries on a Riemannian manifold M, the fixed point set F is nicely behaved. It is known that each component F_α of F is a totally geodesic submanifold of M whose dimension has the same parity as the dimension of M (see e.g., S. Kobayashi, *Fixed points of isometries*, Nagoya Math. J., 13 (1958), 63–68). When M is compact Kähler the isometries are holomorphic transformations and the F_α are compact Kähler submanifolds (which may reduce to points); in particular, as cycles, these components cannot bound in M. This paper is mainly concerned with the structure of the fixed point set in this Kählerian case; however, the use of the complex structure is mainly for convenience; our results also hold for the special type of symplectic manifold in which the fundamental exterior 2-form is harmonic.

As has been pointed out to us by several people, our situation is equivalent to having a toral group acting complex analytically on a compact Kähler M.

Bott [2] has given some important results on the homology of certain homogeneous spaces and the loop space to a group. Our main results, the theorem and corollaries of § 4 can be considered as direct generalizations of the former (see Corollary 3). Our method yields, at the same time, new proofs of his results.

Our proofs are simple applications of another phase of Bott's work, namely his extension of the Morse theory of critical points to functions with "non-degenerate critical manifolds" [3].

The following simple example illustrates the method. Let S_2 be the 2-sphere and let Φ_t be the 1-parameter group of rotations of S_2 about the z axis. The fixed (or stationary) set F of Φ_t consists of the north and south poles, i.e., the places where the velocity vector X vanishes. Now S_2 is a Riemann surface; it has an operator J (multiplication by $(-1)^{\frac{1}{2}}$) that sends any tangent vector into a tangent vector orthogonal to it. The resulting vector field JX gives a flow going from the south pole to the north pole. In fact, $JX = \operatorname{grad} \phi$, where ϕ is a function whose level curves are the circles of latitude and whose critical points consist of a minimum

at the south pole and a maximum at the north pole. We note the important fact that the critical set of ϕ coincides with the fixed set of original group Φ_t. The Morse inequalities then allow us to read off the Betti numbers of S_2 in terms of the fixed set F. In the present paper we show that this situation arises under rather general circumstances.

2. Hypotheses and notation

For all our results the following hypotheses and notation are used without further mention:

(1) M is compact, connected, Kähler with real dimension $n = 2k$.

(2) Φ is a connected 1-parameter group of isometries of M. As mentioned in the introduction (2) is equivalent to

(2') Φ' is a toral group acting complex analytically on M. By the usual averaging process Φ' can be made to act by isometries. One can then extract a 1-parameter group Φ that lies dense on Φ' and we are in situation (2) again. This process is reversible since the group of isometries of M is compact.

The fixed point set F of Φ on M is the set of points of M left fixed by all the transformations of Φ. In the case (2') the fixed set of Φ' coincides with the fixed set of Φ. The components of F are denoted by $F_1, \cdots, F_N$. Each F_α is a compact Kähler submanifold of M.

$b_i(A)$ denotes the ith Betti number for rational coefficients of the space A, while $b_i(A; K)$ denotes the ith Betti number for the coefficient field K.

3. The principal lemmas

LEMMA 1. *If $b_1(M) = 0$, the fixed set F coincides with the non-degenerate critical set of a real C^∞ function ϕ on M.*[2]

PROOF. It is known that the group Φ acts complex analytically on M. Let X be the velocity vector field of the flow on M caused by Φ (since Φ acts by isometries, X is a Killing field). $\theta(X)$, $i(X)$ and d will denote the operations of Lie derivation, interior product (contraction), and exterior derivation respectively, see [5]. These operators, which act on exterior differential forms, are related by the identity

$$\theta(X) = i(X)d + di(X) .$$

We shall apply this identity to the Kähler 2-form ω of M. Since M is Kähler $d\omega = 0$; in fact ω is harmonic. Harmonic forms are invariant

[2] The function in question has arisen in the work of Y. Matsushima, ["Sur la structure du groupe d'homeomorphismes analytiques d'une certaine variete kaehlerienne", Nagoya Math. J., 11 (1957), 145-50] as we learned after submitting this paper.

under connected groups of isometries and so $\theta(X)\omega = 0$, see [11, page 49], hence we may conclude that $di(X)\omega = 0$; i.e., $i(X)\omega$ is a closed 1-form. Since $b_1(M) = 0$ we may integrate this 1-form to get a C^∞ function ϕ on M such that

$$(1) \qquad\qquad d\phi = i(X)\omega .$$

As we are actually using only the real structure of M, ϕ can be taken as real valued.

In terms of real local coordinates $X = X^i \partial/\partial x^i$ and $\omega = w_{ij}\, dx^i \wedge dx^j$; then $i(X)\omega = w_{il}\, X^l\, dx^i$. The critical points of ϕ are where $d\phi = 0$, i.e., where $i(X)\omega = 0$. Since ω is a non-degenerate 2-form, $\det(w_{ij}) \neq 0$ and so the critical points of ϕ are precisely the points where X vanishes, i.e., the points fixed under the group Φ.

The degeneracy of a critical point is determined by the Hessian matrix $\mathscr{H}$ of second partial derivatives of ϕ at the critical point. We have shown in (1) that

$$\frac{\partial \phi}{\partial x_i} = w_{il} X^l$$

and so

$$(2) \qquad\qquad \mathscr{H}_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} = w_{il}\frac{\partial X^l}{\partial x_j}$$

at a critical point (i.e., where $X = 0$).

Let p be a critical point of ϕ. p lies on a compact Kähler submanifold $F(p)$ of critical points (we consider an isolated point as a Kähler submanifold). Let real $\dim F(p) = 2r$, $r \geqq 0$. Let T_p be the tangent space to M at p and let h_p be the subspace of T_p tangent to $F(p)$.

Since Φ leaves p fixed and operates by isometries it induces a 1-parameter group Φ_* of rotations of T_p. Let S be the infinitesimal generator of Φ_*, i.e., $\Phi_*(t) = \exp tS$.

Consider the linear transformation $R \colon T_p \to T_p$ defined by sending the vector $Y = Y^i \dfrac{\partial}{\partial x^i}$ at p into the vector $\dfrac{\partial X^i}{\partial x^j} Y^j \dfrac{\partial}{\partial x^i}$. Extend Y to a vector field, also called Y, in a neighborhood of p. Then since X vanishes at p, we have

$$\frac{\partial X^i}{\partial x^j} Y^j = \frac{\partial X^i}{\partial x^j} Y^j - \frac{\partial Y^i}{\partial x^j} X^j$$

at p. Thus $R(Y) = [Y, X]$. But, see [9]

$$[Y, X] = \lim_{t \to 0} \frac{1}{t} \{\Phi_*(t) - 1\} Y = SY$$

and so $R(Y) = SY$. In terms of matrices then $\mathcal{H}_{ij} = w_{il}S^l_j$ and in terms of linear transformations $\mathcal{H}^i_j = w^i_l S^l_j$. With abuse of notation we may write this as

$$(3) \qquad\qquad\qquad \mathcal{H} = JS$$

where J is the complex structure tensor $J^i_l = w^i_l$ and $J^2 = -1$.

The statement $\theta(X)\omega = 0$ implies $JS = SJ$ (X preserves the complex structure). Thus J and S can be brought to canonical form simultaneously. This means that we can find 2-planes $e_1, \cdots, e_{k-r}$ at p, each invariant under J and S, such that $T_p = e_1 \oplus e_2 \oplus \cdots \oplus e_{k-r} \oplus h_p$ is an invariant orthogonal decomposition. Since S is the infinitesimal generator of a 1-parameter subgroup Φ_* of the rotation group operating on T_p we have the $2k$ by $2k$ matrices

$$
J = \begin{pmatrix} V & & & \\ & V & & 0 \\ & & \ddots & \\ 0 & & & V \end{pmatrix}
\qquad
S = \begin{pmatrix} \Theta_1 & & & & & \\ & \ddots & & 0 & & \\ & & \Theta_{k-r} & & & \\ 0 & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}
$$

where $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\Theta_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}$. θ_i represents the angular velocity of the plane e_i, the orientation being determined by J. For the Hessian we then have from (3)

$$(4) \qquad \mathcal{H} = \mathrm{diag}\,(-\theta_1, -\theta_1, \cdots, -\theta_{k-r}, -\theta_{k-r}, 0, \cdots, 0)$$

The nullity of $\mathcal{H}$ is then $2r$, i.e., the nullity of $\mathcal{H}$ is precisely the dimension of the critical manifold $F(p)$.[3] In the terminology of Bott [3], $F(p)$ is a non-degenerate critical manifold of ϕ. This concludes the proof of the lemma.

COROLLARY. *If $b_1(M) = 0$ and if Φ is not the identity, then the fixed set F is not empty and not connected.*

PROOF. Since ϕ is not constant it has unequal maximum and minimum, A and B respectively. The disjoint level sets $\phi = A$ and $\phi = B$ contain therefore two components of F.

LEMMA 2. *In Lemma 1 we may replace the requirement $b_1(M) = 0$ by the requirement that F be non empty.*

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[3] All θ_i, $1 \leq i \leq k - r$ are non-zero; for if some θ_i were zero the plane e_i would be fixed at T_p and so the geodesic surface through e_i would also be fixed.

PROOF. For the concepts and notation of harmonic forms used in this lemma see [10] and the recent book of A. Weil, "Introduction à l'étude des variétés kaehlériennes," Hermann, 1958. C is the complex structure operator applied to forms.

From the Hodge decomposition theorem we have $i(X)\omega = H[i(X)\omega] + d\phi$ for some function ϕ; here H denotes the harmonic part. Hence we need only show $H[i(X)\omega] = 0$.

Let $\chi = X_i dx^i$ be the "covariant" form for X. From the last part of Theorem 1 of [6] we see that $H[\chi] = 0$; however for completeness we sketch the proof of this statement here. Let h be any harmonic 1-form. Then from the identity $\theta(X)h = i(X)dh + di(X)h$ we again conclude that $di(X)h = 0$, i. e., $i(X)h = $ constant (Bochner's theorem). Since by hypothesis X vanishes somewhere this constant must be 0. Hence $(\chi, h) = \int_M * i(X)h = 0$, and so $H[\chi] = 0$.

The proof of our lemma is concluded by the following observation, for which we gratefully thank S. Kobayashi. $i(X)\omega = C\chi$, hence $H[i(X)\omega] = H[C\chi] = CH[\chi] = 0$.

4. Applications of the critical point theory

The index of a critical manifold F_α, denoted by λ_α, is the number of negative eigenvalues of the Hessian. From (4) we see that in our case λ_α is always even. In Morse's terminology [8] the odd type numbers vanish (for the isolated critical point case).

THEOREM. *If F is non-empty, then*

$$b_i(M; K) = \sum_\alpha b_{i-\lambda_\alpha}(F_\alpha; K)$$

for all i and for coefficient field K either the rationals Q or the integers mod p, Z_p, p prime.

PROOF. The procedure follows that in [4] with the modifications required to discuss critical manifolds rather than isolated critical points. The function ϕ of our lemmas gives rise to a filtration Ω_p of the singular chains of M, the filtration being bounded from below and above. The term E^1 of the resulting spectral sequence is given by $E^1_{p,q} = H_{p+q}(\Omega_p, \Omega_{p-1}; K)$ and Theorem 1 of [3] essentially evaluates $_iE^1 = \sum_p H_i(\Omega_p, \Omega_{p-1}; K)$ as follows

$$\dim \, _iE^1 = \sum_\alpha \dim H_{i-\lambda_\alpha}(F_\alpha; K) \, .$$

We are permitted to use any coefficient field K, rather than only Z_2 as in [3] because in our situation the bundles used in [3] are orientable. That is, the "negative normal bundle" (the sub-bundle of the normal bundle

of F_α corresponding to the negative eigenspaces of the Hessian $\mathcal{H}$) has complex linear subspaces as fibers. This results easily from $\mathcal{H}J = J\mathcal{H}$ which in turn follows immediately from (3) and $JS = SJ$. Since a complex subspace has a natural orientation the bundles in question are orientable.

The Morse-Bott inequalities

$$b_i(M;\ K) \leqq \sum_\alpha b_{i-\lambda_\alpha}(F_\alpha;\ K)$$

then follow from dim $_iE^1 \geqq$ dim $_iE^\infty =$ dim $H_i(M;\ K)$.

On the other hand some work of Floyd [Trans. Amer. Math. Soc., 72 (1952), 138-147] shows that under very general circumstances

$$\sum b_i(M;\ K) \geqq \sum b_i(F;\ K)$$

for $K = Q$ or Z_p. (See also the recent paper of Conner [Mich. Math. J., 5 (1958)]). While this last inequality is stated for finite transformation groups it extends immediately to our toral groups (see § 2) as was pointed out to us by Floyd.

The two inequalities then give our Theorem.

EXAMPLE. Let $M = P_2(C)$ be the complex projective plane with homogeneous complex coordinates $[z_0,\ z_1,\ z_2]$. Let the circle group Φ_t act on M by $[z_0,\ z_1,\ z_2] \rightarrow [z_0,\ e^{it}z_1,\ e^{it}z_2]$. The fixed set F consists of the point $F_1 = [1,\ 0,\ 0]$ and the complex projective line (2-sphere) $F_2 = [0,\ z_1,\ z_2]$. By changing sign if necessary we can choose ϕ so that F_1 is the minimum set and F_2 is the maximum set; i.e. $\lambda_1 = 0$ and $\lambda_2 = 2$. The equalities of the Theorem are easily verified.

COROLLARY 1. *If F is non empty, then*
1°. *F has torsion if and only if M has torsion.*
2°. *$H_{2i+1}(F;\ Z) = 0$, all i if and only if $H_{2i+1}(M;\ Z) = 0$, all i.*

PROOF. Absence of torsion is equivalent to having rational Betti numbers coinciding with mod p Betti numbers for all primes p. 1° then follows immediately from the Theorem. To prove 2° in one direction we note that each F_α is a compact, orientable manifold of even dimension. Thus from Poincaré duality $H_{2i+1}(F_\alpha;\ Z) = 0$ implies that F_α has no torsion. From 1° we conclude that M has no torsion. Now the indices λ_α are all even, hence from the Theorem we conclude that $H_{2i+1}(M, Z) = 0$, all i. The implication in the other direction follows similarly.

COROLLARY 2. *If the fixed points are isolated then M has no torsion and its odd dimensional Betti numbers vanish.*

COROLLARY 3. (Bott). *Let G be a compact semi-simple Lie group, T'*

a toral subgroup and $C(T')$ its centralizer in G. Then $G/C(T')$ has no torsion and its odd dimensional Betti numbers vanish.

PROOF. Borel [1] has shown that the spaces $G/C(T')$ are precisely the Kählerian coset spaces of compact Lie groups, having first Betti number 0. The subgroup $C(T')$ contains a maximal torus T of G. It was shown by Weil, and independently Hopf and Samelson [7] that if one takes a 1-parameter subgroup Φ of T that lies dense on T, the resulting flow on $G/C(T')$ will have isolated fixed points. The flow is isometric with respect to an invariant Kähler metric. Our conclusion then follows from the previous corollary.

REMARK. One can also obtain the "sign" formula for the Betti numbers of G/T, see [4, page 252]. This follows from the fact that the fixed points of Φ lie under the normalisor $N(T)$ of T in G.

EXAMPLE. This example, which was kindly communicated to us by E. Calabi, illustrates that Corollary 2 is not contained in Corollary 3. Let $M' = P_2(C)$ be the complex projective plane and let the circle group Φ' act on M' by $[z_0, z_1, z_2] \to [z_0, e^{it}z_1, e^{2it}z_2]$. The fixed set consists of the three points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, and $p_3 = [0, 0, 1]$. Let M be the space obtained from M' by blowing up the point p_1 into the complex projective line $P_1(C)$ of complex directions at p_1(Hopf σ-process). Since p_1 is fixed under Φ', the induced group Φ'_* rotates $P_1(C)$ onto itself. Thus Φ' augmented by Φ'_* on $P_1(C)$ gives rise to a 1-parameter group Φ of transformations of M. M is still Kähler and Φ still operates analytically. The fixed points on M are the points p_2 and p_3 together with the points on $P_1(C)$ left fixed by Φ'_*. The latter are two in number and correspond to the directions $\overline{p_1p_2}$ and $\overline{p_1p_3}$ at p_1. The space M is not homogeneous.

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